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Refining gear measurement uncertainty calculations by using the Welch-Satterthwaite equation for effective degrees of freedom

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Abstract
When carrying out any form of precision measurement, early consideration should be given to the uncertainty of the measurement results. The primary reference document for evaluation is the "ISO Guide to the expression of uncertainty in measurement". When dealing with variable measurement data on a continuous scale, the structure and mathematical methods will remain similar over different instruments, however the specific sources of error and uncertainty will be dependent upon the instrument and nature of the quantity under study. Various mathematical models are used to calculate uncertainty (such as partial derivatives and the Monte-Carlo method). In gear measurement BS ISO 18653:2003 addresses traceability, calibration intervals, sources of measurement uncertainty or errors including mechanical alignment and drift (among others). Basic instrument checks include environmental factors and methods to evaluate gear uncertainty. The UK National Gear Metrology Laboratory (NGML) utilises the "spreadsheet model" to evaluate measurement uncertainty. The evaluation of gear dimensions defined in ISO 1328-1:2013 requires specific elements of the gear (profile, lead or helix, and pitch) to be considered independently, so a series of spreadsheets are utilised. Since each of the various sources of uncertainty generally have a small number of repeat checks (if any), applying the Welch-Satterthwaite equation allows effective degrees of freedom (v_{eff}) to be calculated for all the sources related to the specific element under study. This will result in a working model which can calculate a coverage factor (k) based on a confidence interval that will compensate for any number of repeat measurements (n) from each source, and therefore result in a more statistically sound outcome.

Keywords: gear metrology, measurement uncertainty, spreadsheet model, welch-satterthwaite, degrees of freedom, *t*-distribution

1. Developing the Uncertainty Budget

When compiling the uncertainty budget for any measurement process, the general sources to be considered include skills, equipment, measurand, procedures, software, calculations and environment. Guidance and case studies are provided by UKAS [1]. In the world of gears, ISO 18653 [2] recommends guidance for measurement strategy and evaluation procedures for estimating the measurement uncertainty with calibrated master gears. When reporting any sources of uncertainty, it is necessary to list the following for each source: units, mean value (u_{95}), distribution type and divisor (*), sensitivity coefficients (c_i) where applicable, and the number of repeated measurements (n) from each source. We can then calculate the individual uncertainty (u_i) for each of these sources as shown in equation (1). When the uncertainties from all sources have been calculated, they can be added in quadrature to find the combined standard uncertainty ($u_{c\,(y)}$). To find the expanded uncertainty (U), we simply multiply the combined standard uncertainty by the coverage factor (k), which is most often two for the 95% confidence interval. The coverage factor used for the uncertainty analysis must be reported on all calibration certificates issued by any UKAS registered laboratory, as it is a requirement of the relavalent ISO standard [3].

$$u_i = \frac{u_{95} \times c_i}{divisor \times \sqrt{n}} \qquad (1)$$

2. Uncertainty types

Sources of uncertainty are defined as either type A or type B. Type A uncertainties are related to random errors, while type B uncertainties have a connection to systematic errors. Random error occurs when repeating the measurement provides randomly different results. If so, the more measurements taken, the higher the chance we can generally expect to get closer to the true value. Systematic error is where the same influence factors affect the results for each of the repeated measurements. In this case, repeating measurements does not affect the quality of the result. Here, other methods are needed to estimate uncertainties due to systematic effects, i.e. different measurements or calculations. Random errors can be revealed as we repeat the measurement. Systematic errors can be revealed when we vary the conditions, whether deliberately or unintentionally. Type A uncertainty is associated with the normal distribution, while type B uncertainty is associated with various other distribution types. The most commonly used distribution for type B uncertainty at the UK National Gear Metrology Laboratory (NGML) is the rectangular distribution.

2.1. Dealing with distributions

The central limit theorem [4] states that the sum of a set of independent random variables will approach a normal distribution as the size of the sample increases, and regardless of the population's original distribution (dist) shape. This theorem assumes that each random variable identifies a source of uncertainty, that no single source or single distribution dominates, and that the sample size is large enough. The coverage factor *k*=2 comes from basic statistical theory which states that plus or minus two measures of the calculated metric of sample standard deviation (σ_{n-1}), will cover approximately 95% of the confidence interval. However, where the number of retests (n) are few (and especially fewer than six which they very often are), it is advisable to modify the coverage factor. Figure 1 shows the probability density functions (PDF) for the normal and *t*-distributions, both of which are symmetrical in shape. Here, both have a mean of zero and the same standard deviation (SD). The PDF for the *t*-distribution is very sensitive when the numbers of tests or degrees of freedom (df or ν) are small. Table 1 shows a partial reconstruction of the *t*-table with the critical values at various confidence intervals. At three degrees of freedom, the sample standard deviation (or X value), would have to be multiplied by 3.18 rather than 2 to determine the 95% confidence interval. At 20 degrees of freedom the value would be 2.09. At infinite degrees of freedom, the two curves would become identical.

Figure 1. Normal and, *t*-distribution (with same mean and SD)

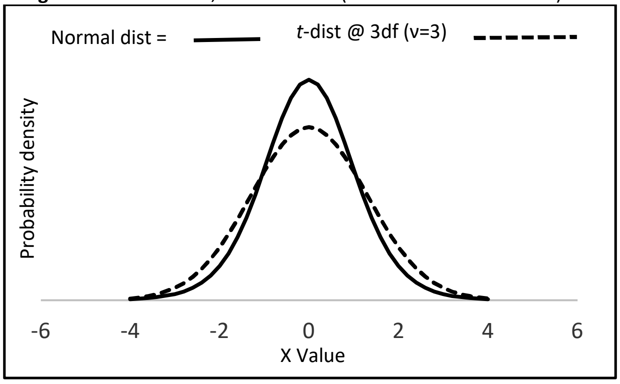

Table 1 Table of critical values for a two tailed *t*-test (up to 6 df)

Degrees of freedom (df)	Confidence Interval 90%	Confidence Interval 95%	Confidence Interval 99%	Confidence Interval 99.5%
1	6.31	12.71	63.7	63.7
2	2.92	4.30	9.92	31.6
3	2.35	**3.18**	5.84	12.90
4	2,13	2.78	4.60	8.61
5	2.02	2.57	4.03	6.86
6	1.94	**2.45*****	3.71	5.96

3. Applying the Welch-Satterthwaite Equation

When dealing with calculating any uncertainty models and especially where retest numbers (n) differ between each uncertainty source, the Welch-Satterthwaite equation (2) can calculate the effective degrees of freedom (ν_{eff}) required.

$$v_{eff} = \frac{u^4_{c(y)}}{\sum_{i=1}^{n} \frac{u^4_{(x_i)}}{v_i}} \tag{2}$$

Here, $u^4_{c\,(y)}$ is the combined standard uncertainty, and $u^4_{(xi)}$ is each individual uncertainty (both raised to the power of 4), while v_i represents the associated individual degrees of freedom from each source. Table 2 represents a typical spreadsheet model [4], with three sources of uncertainty identified as a, b, and c. We first calculate our combined standard uncertainty $u_{c\,(y)}$ (the sum of all individual uncertainties in quadrature), and multiply this value by our defined coverage factor (*k*=2) to calculate our expanded uncertainty (*U*). Next, the Welch-Satterthwaite modifications are applied to calculate v_{eff} for a modified coverage factor *k*. These calculations are shown in table 3.

Table 2 Calculating Combined Standard Uncertainty

Source	Units	Value	Dist*	Divisor	c_i	n	u_i
a	µm	1.000	n	2	1.00	5	0.224
b	µm	0.200	r	1.732	1.00	1	0.115
c	°C	2.000	r	1.732	0.04**	1	0.046
$u_{c\,(y)}$ (the sum of the individual values in quadrature)							0.256
Expanded Uncertainty *U* (*k*=2)							0.51

* n =normal dist with divisor=2, r = rectangular dist and divisor =√3

** calculated by reference to the coefficient of thermal expansion for the specific measurand material.

Table 3 Coverage factor by effective degrees of freedom

Source	Value	Dist	Divisor	c_i	n	u_i	u_i^4/n
a	1.000	n	2	1.00	5	0.224	0.0005
b	0.200	r	1.732	1.00	1	0.115	0.0002
c	2.000	r	1.732	0.04	1	0.046	0.0000
e Combined Standard Uncertainty $u_{c\,(y)}$						0.256	
f Modified Standard Uncertainty $u^4_{c\,(y)}$						0.0043	
g Sum of Values (for a, b, and c)							0.0007
$\nu_{eff} = f/g$							6.143
Modified Coverage Factor k based upon ν_{eff}							2.447
Modified Expanded Uncertainty *U* (*k*=2.45)							0.63 µm

4. Observations

The modified value of 2.447 for coverage factor *k* was found by applying the function "=TINV(x, y)" in Microsoft Excel®. This function returns the inverse of the yth percentile of the x values. In our specific case, x represents the 95% confidence interval (0.05), and y is our calculated value for ν_{eff} rounded to the nearest whole number (6). This rounded value is shown in table 1 (***). We find our modified expanded uncertainty *U* by multiplying our modified coverage factor *k* by the combined standard uncertainty $u_{c\,(y)}$. Normal practice is to maintain an appropriate number of decimal places at the spreadsheet stage, but to round the final uncertainty results to two decimal places.

5. Conclusions

By applying the Welch-Satterthwaite equation modifications we can offer a more realistic value for the coverage factor for any number of uncertainty sources with any value of "n". Initial studies at the NGML have shown interesting results and further exploratory trials are being carried out.

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